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Imploding Shocks in a Non-Ideal Medium

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Abstract. Self-similar solutions of the second kind for the unsteady one-dimensional flow behind converging spherical and cylindrical strong shocks in a non-ideal medium are studied. The equation of state of the medium is assumed to be in the form of the Mie-Gruneisen type. A simple numerical technique is developed to find the similarity exponent. Detailed studies are carried out for a different non-ideal medium such as a dusty gas, condensed matter and c,s medium. The solutions for an ideal gas are recovered as a particular case. A comparative study of the numerical and some approximate solutions is also made.

1. Introduction

Study of imploding shocks is of great importance in achieving the higher compression of the medium in which it propagates. The motion of converging spherical strong shock waves in a perfect gas has been studied by many authors [1–6]. The converging spherical strong shock wave in a perfect gas, analyzed by Guderley [1], is one of the first examples of a class of solutions known as self-similar solutions of the second kind [3]. Guderley [1] demonstrated the existence of self-similar solutions for the shock wave propagating in the vicinity of the center of convergence. Stanyukovich [2] has given an approximate method which yields the value of the similarity exponent which is quite close to the correct value. This value was used as an initial guess and then refined. Fujimoto and Mishkin [5] observed that the pressure behind the shock wave propagating in a perfect gas increases and attains a maximum value. This concept of single-peak pressure behind the shock front leads to an analytical determination of the similarity exponent. A comparative study of these two techniques is given by Yousaf [8]. A power-series solution for the converging strong shock waves in a perfect gas was developed by Hafner [4]. Ramu and Ranga Rao [7] have presented the self-similar solutions for strong shock waves in a non-ideal medium satisfying the equation of state of the Mie-Gruneisen type. But they have put a condition on the Mie-Gruneisen coefficient so that the phase-plane analysis could be applied for solving the problem.

In this paper a study is made of the propagation of an imploding shock in a uniform medium at rest whose equation of state is of the Mie-Gruneisen type. Basic equations are transformed into a set of first-order ordinary differential equations by similarity transformations. The shock is assumed to be strong. A more general Mie-Gruneisen coefficient $\Gamma(\rho/\rho_0)$ where ρ, ρ_0 denote the density of the medium and initial density, respectively, is used in the equation of the state of the medium and the similarity exponent and also the flow variables for three physically meaningful Mie-Gruneisen coefficients are found. The numerical method used to calculate the similarity exponent is a one-parameter iterative process which is simpler and also applicable for finding the self-similar solutions of the second kind for shock waves due to impulsive load [14] in a non-ideal medium with any Mie-Gruneisen coefficient.

Similarity exponents for converging shocks in perfect gases were evaluated analytically by Stanyukovich [2], using a phase-plane analysis. Since it is not possible to reduce the system of ordinary differential equations governing the flows behind the shock waves in a non-ideal medium into an equation in the phase plane, this analysis is found to be inapplicable. The analytical method of Fujimoto and Mishkin [5] to determine the similarity exponent seems to be inapplicable for the non-ideal medium, even though there exists a peak in the pressure profile near the shock front. However the approximate method developed by Chester, Chisnell and Whitham which is known as CCW rule [6] (page 271) is found to be effective in calculating the similarity exponent.

2. Basic Equations

The basic conservation equations of mass, momentum and energy governing adiabatic flow in Eulerian coordinates are

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \nu \frac{\rho u}{x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{p}\frac{\partial p}{\partial x} = 0, \tag{2}$$

$$\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x} - \frac{p}{\rho}\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right]\log(\rho/\rho_0) = 0, \tag{3}$$

where $\nu = 1, 2$ for cylindrical and spherical flows, ρ, u, p, e are density, velocity, pressure and the specific internal energy of the medium, respectively. The equation of state for the medium is assumed to be

$$p = \rho e \Gamma\left(\frac{\rho}{\rho_0}\right), \tag{4}$$

where $\Gamma(\rho/\rho_0)$ is the Mie-Gruneisen coefficient. The shock is assumed to be strong and propagating according to the power law $X(t) \propto (1 - t/t_c)^\alpha$, where t_c denotes the collapse time, X is the shock position at the time t and α is the similarity exponent to be found. At the shock front, $x = X(t)$, the boundary conditions are given by the Rankine-Hugoniot jump conditions

$$u_1 = \left(1 - \frac{\rho_0}{\rho_1}\right)\dot{X}, \tag{5}$$

$$p_1 - p_0 = \rho_0 u_1 \dot{X}, \tag{6}$$

$$e_1 - e_0 = \frac{1}{2}u_1^2 + \frac{p_0}{\rho_1}\left(1 - \frac{\rho_0}{\rho_1}\right), \tag{7}$$

where the subscripts 1 and 0 denote the quantities just behind and ahead of the shock front, respectively, and $\dot{X} = \mathrm{d}X/\mathrm{d}t$ is the shock speed. The strong shock conditions ($p_1 \gg p_0$) for the present problem can be written as, using (5)–(7),

$$\rho_1 = \rho_0 \beta, \tag{8}$$

$$u_1 = \left(1 - \frac{1}{\beta}\right)\dot{X}, \tag{9}$$

$$p_1 = \rho_0\left(1 - \frac{1}{\beta}\right)\dot{X}^2, \tag{10}$$

where β is the compression across the shock wave. Using the strong shock condition (5)–(7) and Eq. (4), we can derive the relation

$$\Gamma(\beta)(\beta - 1) = 2. \tag{11}$$

This equation enables us to calculate β, once $\Gamma(\rho/\rho_0)$ is known.

We make use of the similarity transformations

$$\begin{aligned} \rho(x,t) &= \rho_0 G(\xi), \\ u(x,t) &= \dot{X}V(\xi), \\ p(x,t) &= \rho_0\dot{X}^2 H(\xi), \\ \xi &= \frac{x}{X(t)}, \end{aligned} \tag{12}$$

and reduce the partial differential equations (1)–(3) to a system of ordinary differential equations.

$$(\xi - V)\frac{G'}{G} - V' = \frac{\nu V}{\xi}, \tag{13}$$

$$(\xi - V)V' - \frac{H'}{G} = \lambda V, \tag{14}$$

$$\frac{H'}{H} - \Phi(G)\frac{G'}{G} = \frac{2\lambda}{(\lambda - V)}, \tag{15}$$

where

$$\Phi(G) = 1 + \Gamma(G) + \frac{G}{\Gamma(G)}\frac{\mathrm{d}\Gamma(G)}{\mathrm{d}G}, \qquad \lambda = 1 - \frac{1}{\alpha},$$

and V, G, H are the non-dimensional functions of the similarity variable ξ and a prime denotes the derivative with respect to ξ.

The transformed strong shock conditions are

$$\begin{aligned} G(1) &= \beta, \\ V(1) &= 1 - \frac{1}{\beta}, \\ H(1) &= 1 - \frac{1}{\beta}. \end{aligned} \tag{16}$$

The transformed equations (13)–(15) are nonlinear ordinary differential equations and the analytical solution for such a system of equations is not possible. These equations are therefore solved numerically.

Using the fact that the relative particle velocity of the shock front is subsonic, i.e.

$$(u_1 - \dot{X})^2 < C_1^2, \qquad C_1^2 = \Phi(\beta)\frac{p_1}{\rho_1}, \tag{17}$$

where C_1 is the sound speed at the shock, we can derive the condition on $\Gamma(G)$ by using (11) as

$$\frac{\mathrm{d}}{\mathrm{d}G}\left[\frac{2}{\Gamma(G)}\right]_{G=\beta} < 1. \tag{18}$$

This condition is also derived in [11] for a different purpose.

3. Numerical Method

In this section we shall show how to determine the similarity exponent α in solving Eqs. (13)–(15). In order to do this, we first write the equations in the form

$$G'(\xi) = \frac{\Delta_1}{\Delta}, \qquad V'(\xi) = \frac{\Delta_2}{\Delta}, \qquad H'(\xi) = \frac{\Delta_3}{\Delta}, \tag{19}$$

where

$$\begin{aligned}
\Delta &= \frac{1}{G^2 H}[G(\xi - V)^2 - H\Phi(G)],\\
\Delta_1 &= \frac{\nu V(\xi - V)}{H} + \left[\frac{\lambda V}{H} + \frac{2\lambda}{G(\xi - V)}\right],\\
\Delta_2 &= \frac{\nu\Phi(G)V}{\xi G^2} + \frac{(\xi - V)}{G}\left[\frac{\lambda V}{H} + \frac{2\lambda}{G(\xi - V)}\right],\\
\Delta_3 &= \frac{\nu(\xi - V)\Phi(G)V}{G\xi} + \frac{2\lambda(\xi - V)}{G} + \frac{\lambda V\Phi(G)}{G}.
\end{aligned} \tag{20}$$

For a given Mie-Gruneisen coefficient, the compression β is first calculated from Eq. (11). Then we start integrating the system (19) from $\xi = 1$ with the conditions (16) by a Runge-Kutta method with some arbitrary value of α. The correct value of α is determined by employing the one-parameter (α) iterative procedure such that all the determinants $\Delta, \Delta_1, \Delta_2$ and Δ_3 become zero simultaneously at the same value of ξ in the domain of interest $1 < \xi < \infty$, which is the singular point of the system. For this correct value of α, we get the final non-singular solutions. This method is found to be very effective in getting the self-similar solutions of the second kind for shock waves due to an impulsive load [14] in a non-ideal medium with any Mie-Gruneisen coefficient.

4. Approximate Method

In this section we apply the approximate method developed by Chester, Chisnell and Whitham [6] (page 271) which is known as the CCW rule to calculate the similarity exponent. By this rule the differential equation for p_1 in terms of X is given by

$$\frac{\mathrm{d}p_1}{\mathrm{d}X} + \rho_1 a_1 \frac{\mathrm{d}u_1}{\mathrm{d}X} + \frac{\nu}{X}\frac{\rho_1 a_1^2 u_1}{u_1 + a_1} = 0, \quad a_1^2 = \frac{p_1}{\rho_1}\Phi(\beta), \tag{21}$$

where a_1 is the sound speed immediately behind the shock. Using the strong shock conditions (8)–(10), we may reduce Eq. (21) to

$$\frac{X\ddot{X}}{\dot{X}^2} + \frac{\nu}{n} = 0, \tag{22}$$

where

$$n = \frac{1}{\Phi(\beta)}\left[1 + \sqrt{\frac{\Phi(\beta)}{\beta - 1}}\right]\left[2 + (\beta - 1)\sqrt{\frac{\Phi(\beta)}{\beta - 1}}\right]. \tag{23}$$

Solving Eq. (22), we get

$$X \propto \left(1 - \frac{t}{t_c}\right)^{\alpha}, \quad \alpha = \frac{n}{n + \nu}. \tag{24}$$

For a given $\Gamma(G)$, we first calculate β from Eq. (11). Then we calculate α from Eqs. (23) and (24). This value of α as obtained by the CCW rule is denoted by α_c.

5. Results

Numerical results are obtained in the cases $\nu = 1, 2$ for the three physically meaningful Mie-Gruneisen coefficients $\Gamma(G)$ which are given below. A double precision Fortran-77 program on a CMC-Cyber computer was employed in obtaining these results.

(i) The equation of state of the dusty gas (see Pai [9]) can be shown to be of the Mie-Gruneisen type by taking

$$\Gamma(G) = \frac{K - 1}{1 - Z_0 G}, \tag{25}$$

where Z_0 is the initial volume fraction of the solid particles in the dusty gas and K is the ratio of the specific heats of the dusty gas. Using the relation (11) and the equation (25) we find that the expression for β is given by

$$\beta = \frac{K + 1}{K + 2Z_0 - 1}. \tag{26}$$

The values of the similarity exponent obtained numerically (α_n) and also by the approximate method (α_c), for $K = 1.4$ and different values of Z_0 are given in Table 1. The non-dimensional flow variables for $K = 1.4$ and $Z_0 = 0.04$ are shown in fig. 1. It is observed that when Z_0 increases, α decreases for the same value of K (see Fig. 2).

(ii) The Mie-Gruneisen coefficient for condensed matter [10] is

$$\Gamma(G) = \frac{2}{3} + \left(\Gamma_0 - \frac{2}{3}\right)\frac{G_m^2 + 1}{G_m^2 + G^2}G, \tag{27}$$

where $\Gamma_0 = \Gamma(1)$ and G_m are the parameters which are usually determined from experiment. The parameter Γ_0 is equal to 1.78 for iron, 2.02 for coper and 2.19 for aluminum, while G_m lies in the range between 0.5 and 0.8. For the given values of Γ_0 and G_m, we can evaluate the compression β from (11) and (27), using Newton-Raphson's iterative method. The similarity

Table 1. Similarity exponents for a dusty gas when $K = 1.4$

Z_0	β	$\nu = 1$		$\nu = 2$	
		α_n	α_c	α_n	α_c
0.000	6.0000	0.83533	0.83537	0.71717	0.71729
0.001	5.9702	0.83464	0.83483	0.71605	0.71649
0.005	5.8537	0.83194	0.83270	0.71157	0.71335
0.010	5.7143	0.82858	0.83007	0.70598	0.70950
0.020	5.4545	0.82196	0.82494	0.69480	0.70204
0.040	5.0000	0.80912	0.81518	0.67259	0.68802
0.060	4.6154	0.79684	0.80603	0.64968	0.67508
0.080	4.2857	0.78512	0.79743	0.62844	0.66311
0.100	4.0000	0.77404	0.78934	0.60533	0.65199

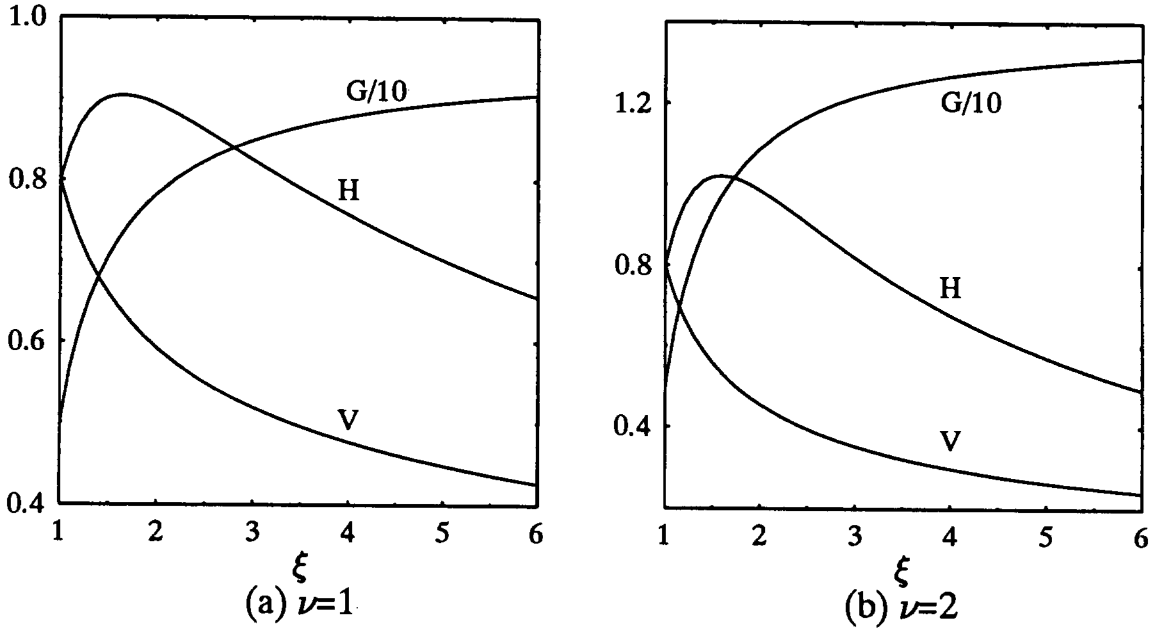

Figure 1. Non-dimensional flow variables for a dusty gas for $K = 1.4$, $Z_0 = 0.04$.

exponent for $\Gamma_0 = 2/3$, corresponding to a perfect gas case with $\gamma = 5/3$, is also calculated. The results for $\Gamma_0 = 1.78, 2.02, 2.19$ and different values of G_m are shown in Table 2 and the non-dimensional velocity, pressure and density for $\Gamma_0 = 1.78$ and $G_m = 0.5$ are plotted in Fig. 3. Anisimov and Kravchenko [11] have used this Mie-Gruneisen coefficient (27) to solve the impulsive load problem in condensed matter, using a two-parameter iterative procedure. This problem with different $\Gamma(G)$ was solved by Patel and Ranga Rao [14] by employing a one-parameter iterative procedure.

(iii) Another Mie-Gruneisen coefficient taken into consideration is (inspired by the Walsh equation of state [12])

$$\Gamma(G) = \frac{\Gamma_0}{G}, \tag{28}$$

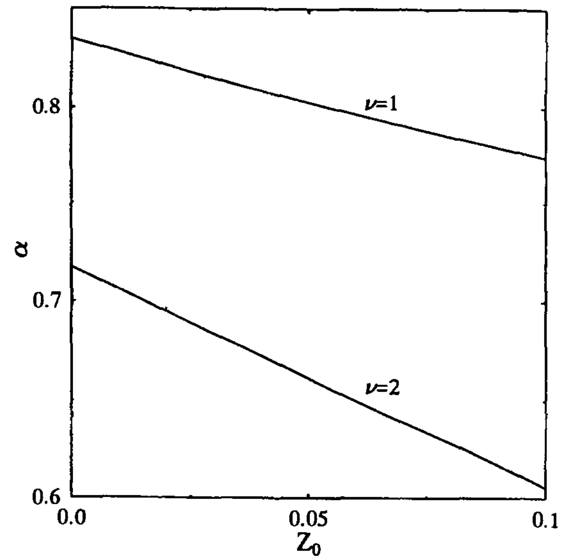

Figure 2. Similarity exponents for a dusty gas for $K = 1.4$.

Table 2. Similarity exponents for condensed matter.

Γ_0	G_m	β	$\nu = 1$		$\nu = 2$	
			α_n	α_c	α_n	α_c
1.78	0.50	2.722	0.81180	0.81026	0.68420	0.68204
	0.65	2.617	0.80960	0.80756	0.68146	0.67723
	0.80	2.506	0.80648	0.80348	0.67740	0.67150
2.02	0.50	2.525	0.80880	0.80665	0.68050	0.67595
	0.65	2.420	0.80570	0.80307	0.67646	0.67094
	0.80	2.312	0.80168	0.79858	0.67120	0.66970
2.19	0.50	2.402	0.80630	0.80376	0.67728	0.67190
	0.65	2.299	0.80266	0.79964	0.67248	0.66619
	0.80	2.199	0.79815	0.79467	0.66657	0.65930
$\Gamma = 2/3$	-	4.000	0.81569	0.81604	0.68840	0.68925

Table 3. Similarity exponents for $\Gamma(G) = \Gamma_0/G$.

Γ_0	β	$\nu = 1$		$\nu = 2$	
		α_n	α_c	α_n	α_c
3.0	3.000	0.85880	0.85355	0.75490	0.74452
4.0	2.000	0.80945	0.80474	0.68289	0.67327
5.0	1.667	0.78587	0.78033	0.65128	0.63979
6.0	1.500	0.77245	0.76568	0.63440	0.62033
7.0	1.400	0.76381	0.75592	0.62420	0.60761
8.0	1.333	0.75782	0.74895	0.61739	0.59865
9.0	1.285	0.75300	0.74371	0.61280	0.59199
10.0	1.250	0.75000	0.73965	0.60890	0.58686

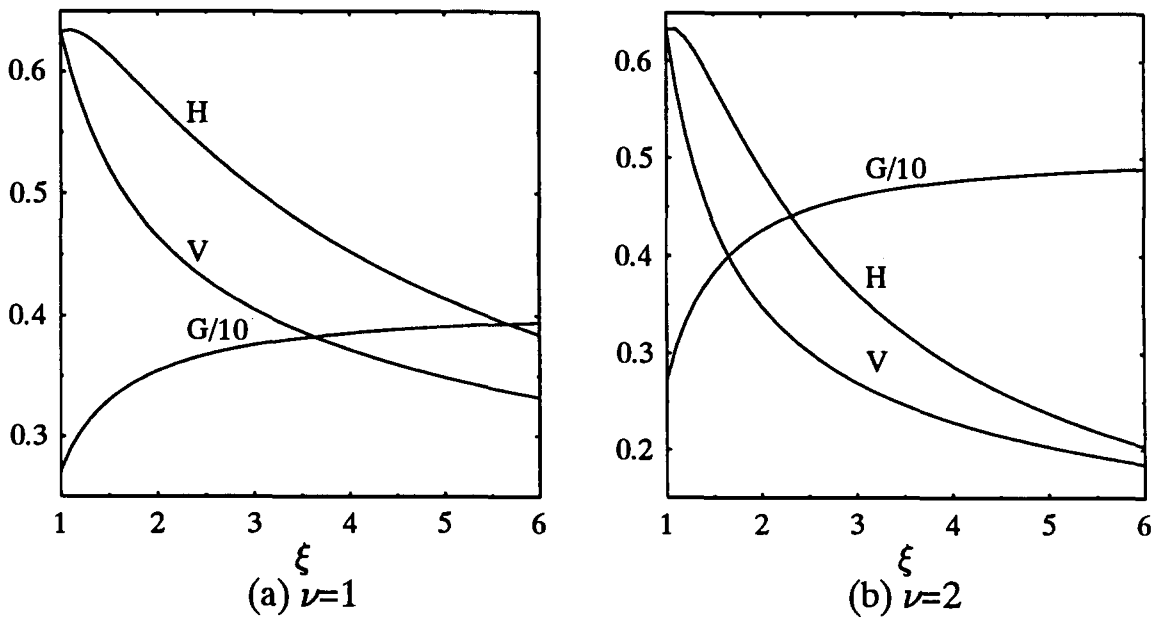

Figure 3. Non-dimensional flow variables for condensed matter for $\Gamma_0 = 1.78, G_m = 0.5$.

where Γ_0 is a known parameter. In this case β can be calculated directly from the relations (11) and (28) as

$$\beta = \frac{\Gamma_0}{\Gamma_0 - 2}. \tag{29}$$

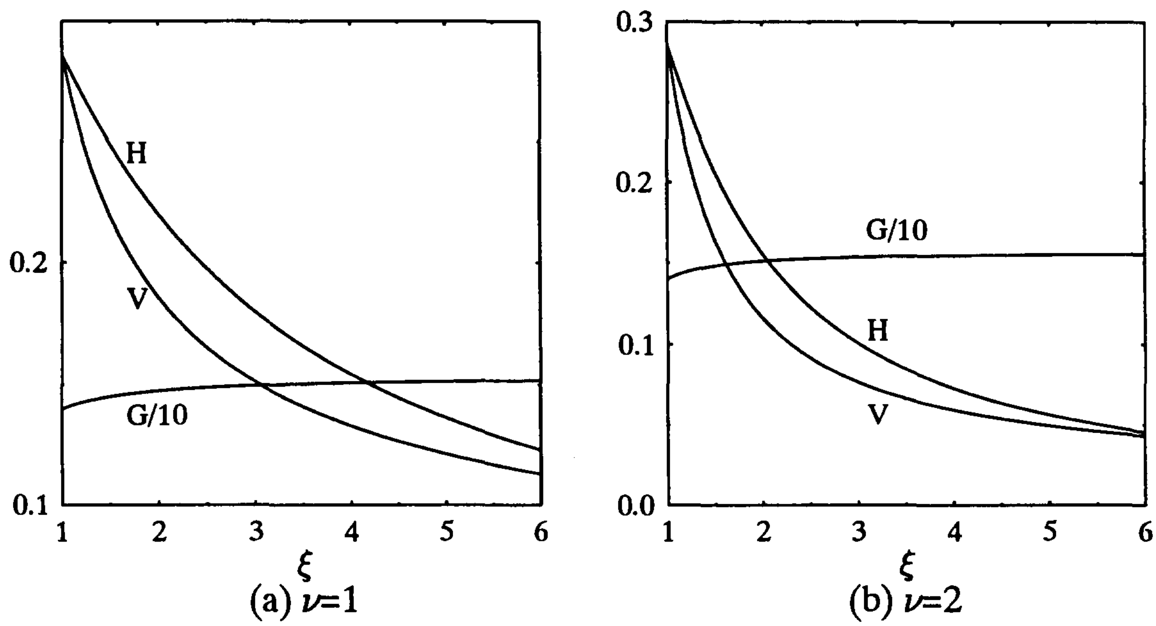

Figure 4. Non-dimensional flow variables for $\Gamma(G) = \Gamma_0/G, \Gamma_0 = 5$.

Here it is interesting to note that for the Walsh equation of state [12]

$$B_s = \frac{p + p_e}{A\rho}, \tag{30}$$

where B_s is the adiabatic bulk modulus, p_e and A are constants. The sound speed C is given by $(C^2\rho = B_s)$

$$C^2 = \frac{p + p_e}{\rho}\frac{\Gamma_0}{G}, \qquad A = \frac{1}{\Gamma_0 \rho_0}. \tag{31}$$

Now if we assume the equation of state to be of a modified form of the Mie-Gruneisen type, *i.e.* $p + p_e = \rho e \Gamma(G)$, then it is easy to show that

$$C^2 = \frac{p + p_e}{\rho}\Phi(G). \tag{32}$$

Equating the two expressions for C^2 from (31) and (32), we get an ordinary differential equation for $\Gamma(G)$ which on integrating with the condition $\Gamma(1) = \Gamma_0$ yields the same form for $\Gamma(G)$ as given by Eq. (28). The advantage of taking this type of equation of state is that it is consistent with the experimentally observed linear relation between the shock speed $\dot{X}$ and particle speed behind the shock u_1, namely $\dot{X} = c + su_1$, where c is a constant related to p_e and s is a constant related to A or Γ_0. These relations can be shown to be

$$\Gamma_0 = 4s, \qquad p_e = \frac{\rho_0 c^2}{4s}. \tag{33}$$

The medium in which the linear relation $\dot{X} = c + su_1$ is satisfied, is the well-known c, s medium [13]. The values of α for different values of Γ_0 are given in Table 3. The non-dimensional flow variables behind the shock wave for $\Gamma_0 = 7$ are shown in Fig. 4. Using the Mie-Gruneisen coefficient given by (28) we recovered the solutions of the problem discussed in [13] for $\Gamma_0 = 5, \nu = 2$. In a perfect gas, for which $\Gamma(G) = \gamma - 1$, the values of α for different values of γ are calculated and they are found to be in close agreement with the results obtained by Guderley.

This numerical technique is applicable for the Mie-Gruneisen coefficient used by Ramu and Ranga Rao [7] and also for any coefficient satisfying the condition (18).

6. Concluding Remarks

In this paper self-similar solutions of the second kind for the unsteady spherical and cylindrical flow behind converging strong shocks propagating in a non-ideal medium at rest are obtained. The equation of state of the medium is assumed in a form so that self similar solutions do exist. A simple numerical iterative method is used to determine the similarity exponent. Also, an approximate method is employed to derive an analytical expression for the similarity exponent. A comparative study between these results shows very good agreement. The classical solution for the perfect-gas problem [1] is recovered as a special case.

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